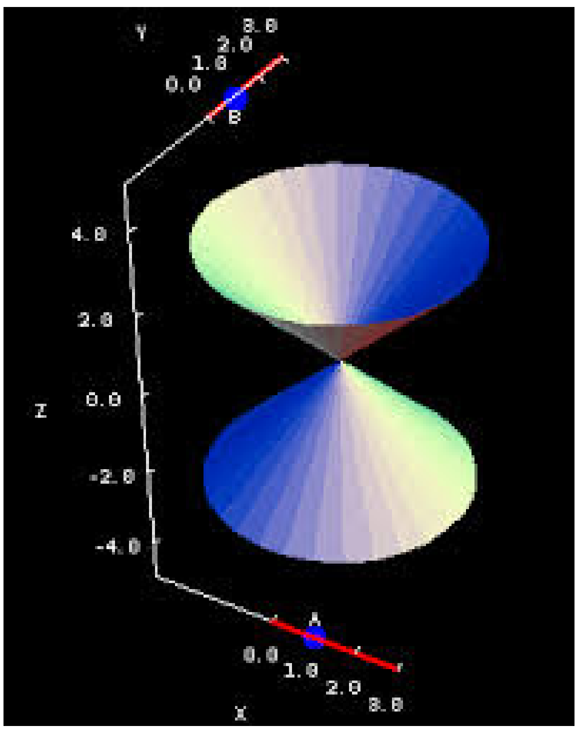
New work of Castillo, Duarte, Leyton-Avarez, and Liendo

Introduction

A recent preprint of these authors , *Nash blowup fails to resolve singularities in dimensions four and higher* has appeared which deserves discussion.

Let me write for the non-mathematical reader to start with. The real double cone is very familiar classically



and it is quite normal to think of it as a cylinder which was embedded inefficiently. The inverse image of the singular point is a circle, and real analytically we don't care whether it is a real projective line, or a unit circle.

That is, someone is justified to think "Oh, a double cone is just an immersed cylinder."

The new paper is the first situation, historically, where that idea can't be repaired just by iterating. Nash (and others) thought singularities in mathematical models aren't meaningful, that they come from inefficient embeddings of natural smooth manifolds. That they aren't really there.

The paper says that the (single!) cone spanned by the four standard basis elements and $[2, 3, -2, -1], [1, 3, -1, -1]$ includes $[1, 2, -1, 0]$, that all the integer points of the cone are positive linear combinations of these seven.

I have to quickly say, the object being described by the lattice polyhedron is a non-compact toric variety, an intro to these things is here on this website *first introduction to toric geometry*.

Now a comment for some with more expertise, that (even if it doesn't come like this one from a lattice polygon) the correponding set for the first Nash transform can come from choosing any number k which has to satisfy $k \geq 2$ and taking sums of $n + k$ elements which are affinely general (a.g.), meaning not contained in an $n - 1$ dimensional hyperplane (which needn't go through the origin). For intance in the plane, $(a, b), (c, d), (e, f)$ is affinely general if $det \begin{pmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{pmatrix} \neq 0$. and then any set containing these three also is. Taking $n = 4, k = 2$ the vertex of the Nash polyhedron they refer to is this a.g. sum of $4 + 2 = 6$

$$(0, 0, 0, 0) + (0, 0, 0, 0) + (1, 0, 0, 0) + (0, 1, 0, 0) + (0, 0, 1, 0) + (2, 3, -2, -1)$$
$$= (3, 4, -1, -1)$$

Their claim is that an affine isomorphism of $\mathbb{Z}^4$ sends their cone to the Nash cone based at this point.

Their result

The original semigroup generated by their seven elements meets this new cone at an ideal of the semigroup, and although the Nash polyhedron is made from a.g. sums of six, the ideal is generated by a.g. sums of five elements from the 8 element set comprising zero with their seven generators (see the def of a in the script). The ring element one would multiply to get an arbitrary element of the ideal is vector number six; and just as when you can include more to $\{(a,b),(c,d),(e,f)\}$ we can include this one more.

Here is a javascript (using a function called det for determinant)

```
function minus(x){return [-x[0],-x[1],-x[2],-x[3] ]    }
function plus(x,y){return [x[0]+y[0],x[1]+y[1],x[2]+y[2],x[3]+y[3]] }
var a=[[0,0,0,0],[1,0,0,0],[0,1,0,0],[0,0,1,0],
                           [0,0,0,1],[2,3,-2,-1],[1,3,-1,-1],[1,2,-1,0]]

function J(){
var res=[]
for(var i=0;i<a.length;i++){
for(var j=i+1;j<a.length;j++){
for(var k=j+1;k<a.length;k++){
for(var l=k+1;l<a.length;l++){
for(var l2=l+1;l2<a.length;l2++){
var v=a[i].slice(0);v.push(1)
var w=a[j].slice(0);w.push(1)
var u=a[k].slice(0);u.push(1)
var p=a[l].slice(0);p.push(1)
var q=a[l2].slice(0);q.push(1)
var mat=[v,w,u,p,q]
if(det(mat)!=0){for(var m=0;m<a.length;m++){
var ssum=plus(a[m], plus(a[i],plus(a[j],plus(a[k],plus(a[l],a[l2])))))
res.push(ssum)
}}}}}
return res
}
var JJ=J()
var vertex=[3,4,-1,-1]
```

```
var JJ2=[]
for(var i=0;i<JJ.length;i++){JJ2[i]=plus(JJ[i],minus(vertex))}
console.log(JSON.stringify(JJ2))
```


```
[[-2,-3,2,2],[-1,-3,2,2],[-2,-2,2,2],[-2,-3,3,2],[-2,-3,2,3],[0,0,0,1],[-1,0,1,1],
[-1,-1,1,2],[0,0,0,0],[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1],[2,3,-2,-1],[1,3,-1,-1],
[1,2,-1,0],[-1,0,1,0],[0,0,1,0],[-1,1,1,0],[-1,0,2,0],[-1,0,1,1],[1,3,-1,-1],
[0,3,0,-1],[0,2,0,0],[0,0,-1,1],[1,0,-1,1],[0,1,-1,1],[0,0,0,1],[0,0,-1,2],[2,3,-3,0],
[1,3,-2,0],[1,2,-2,1],[-1,0,0,1],[0,0,0,1],[-1,1,0,1],[-1,0,1,1],[-1,0,0,2],
[1,3,-2,0],[0,3,-1,0],[0,2,-1,1],[-1,-1,0,2],[0,-1,0,2],[-1,0,0,2],[-1,-1,1,2],
[-1,-1,0,3],[1,2,-2,1],[0,2,-1,1],[0,1,-1,2],[1,3,-2,-1],[2,3,-2,-1],[1,4,-2,-1],
[1,3,-1,-1],[1,3,-2,0],[3,6,-4,-2],[2,6,-3,-2],[2,5,-3,-1],[1,2,-2,0],[2,2,-2,0],
[1,3,-2,0],[1,2,-1,0],[1,2,-2,1],[3,5,-4,-1],[2,5,-3,-1],[2,4,-3,0],[0,2,-1,0],
[1,2,-1,0],[0,3,-1,0],[0,2,0,0],[0,2,-1,1],[2,5,-3,-1],[1,5,-2,-1],[1,4,-2,0],
[0,-1,0,1],[1,-1,0,1],[0,0,0,1],[0,-1,1,1],[0,-1,0,2],[2,2,-2,0],[1,2,-1,0],
[1,1,-1,1],[-1,-1,1,1],[0,-1,1,1],[-1,0,1,1],[-1,-1,2,1],[-1,-1,1,2],[1,2,-1,0],
[0,2,0,0],[0,1,0,1],[-1,-2,1,2],[0,-2,1,2],[-1,-1,1,2],[-1,-2,2,2],[-1,-2,1,3],
[1,1,-1,1],[0,1,0,1],[0,0,0,2],[1,1,-1,0],[2,1,-1,0],[1,2,-1,0],[1,1,0,0],[1,1,-1,1],
[3,4,-3,-1],[2,4,-2,-1],[2,3,-2,0],[0,1,0,0],[1,1,0,0],[0,2,0,0],[0,1,1,0],
[0,1,0,1],[2,4,-2,-1],[1,4,-1,-1],[1,3,-1,0],[1,2,-2,0],[2,2,-2,0],[1,3,-2,0],
[1,2,-1,0],[1,2,-2,1],[3,5,-4,-1],[2,5,-3,-1],[2,4,-3,0],[1,1,-2,1],[2,1,-2,1],
[1,2,-2,1],[1,1,-1,1],[1,1,-2,2],[3,4,-4,0],[2,4,-3,0],[2,3,-3,1],[0,1,-1,1],
[1,1,-1,1],[0,2,-1,1],[0,1,0,1],[0,1,-1,2],[2,4,-3,0],[1,4,-2,0],[1,3,-2,1],
[2,4,-3,-1],[3,4,-3,-1],[2,5,-3,-1],[2,4,-2,-1],[2,4,-3,0],[4,7,-5,-2],
[3,7,-4,-2],[3,6,-4,-1],[-1,0,0,1],[0,0,0,1],[-1,1,0,1],[-1,0,1,1],[-1,0,0,2],
[1,3,-2,0],[0,3,-1,0],[0,2,-1,1],[-2,0,1,1],[-1,0,1,1],[-2,1,1,1],[-2,0,2,1],
[-2,0,1,2],[0,3,-1,0],[-1,3,0,0],[-1,2,0,1],[-2,-1,1,2],[-1,-1,1,2],[-2,0,1,2],
[-2,-1,2,2],[-2,-1,1,3],[0,2,-1,1],[-1,2,0,1],[-1,1,0,2],[0,3,-1,-1],[1,3,-1,-1],
[0,4,-1,-1],[0,3,0,-1],[0,3,-1,0],[2,6,-3,-2],[1,6,-2,-2],[1,5,-2,-1],[0,2,-1,0],
[1,2,-1,0],[0,3,-1,0],[0,2,0,0],[0,2,-1,1],[2,5,-3,-1],[1,5,-2,-1],[1,4,-2,0],
[-1,2,0,0],[0,2,0,0],[-1,3,0,0],[-1,2,1,0],[-1,2,0,1],[1,5,-2,-1],[0,5,-1,-1],
[0,4,-1,0],[0,2,-1,0],[1,2,-1,0],[0,3,-1,0],[0,2,0,0],[0,2,-1,1],[2,5,-3,-1],
[1,5,-2,-1],[1,4,-2,0],[0,1,-1,1],[1,1,-1,1],[0,2,-1,1],[0,1,0,1],[0,1,-1,2],
[2,4,-3,0],[1,4,-2,0],[1,3,-2,1],[-1,1,0,1],[0,1,0,1],[-1,2,0,1],[-1,1,1,1],
[-1,1,0,2],[1,4,-2,0],[0,4,-1,0],[0,3,-1,1],[1,4,-2,-1],[2,4,-2,-1],[1,5,-2,-1],
[1,4,-1,-1],[1,4,-2,0],[3,7,-4,-2],[2,7,-3,-2],[2,6,-3,-1],[0,0,0,1],[1,0,0,1],
[0,1,0,1],[0,0,1,1],[0,0,0,2],[2,3,-2,0],[1,3,-1,0],[1,2,-1,1],[-1,0,1,1],
[0,0,1,1],[-1,1,1,1],[-1,0,2,1],[-1,0,1,2],[1,3,-1,0],[0,3,0,0],[0,2,0,1],
[-1,-1,1,2],[0,-1,1,2],[-1,0,1,2],[-1,-1,2,2],[-1,-1,1,3],[1,2,-1,1],
[0,2,0,1],[0,1,0,2],[1,2,-1,0],[2,2,-1,0],[1,3,-1,0],[1,2,0,0],[1,2,-1,1],
```

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[3,5,-3,-1],[2,5,-2,-1],[2,4,-2,0],[0,2,0,0],[1,2,0,0],[0,3,0,0],[0,2,1,0],
[0,2,0,1],[2,5,-2,-1],[1,5,-1,-1],[1,4,-1,0],[1,3,-2,0],[2,3,-2,0],[1,4,-2,0],
[1,3,-1,0],[1,3,-2,1],[3,6,-4,-1],[2,6,-3,-1],[2,5,-3,0],[1,2,-2,1],[2,2,-2,1],
[1,3,-2,1],[1,2,-1,1],[1,2,-2,2],[3,5,-4,0],[2,5,-3,0],[2,4,-3,1],[2,5,-3,-1],
[3,5,-3,-1],[2,6,-3,-1],[2,5,-2,-1],[2,5,-3,0],[4,8,-5,-2],[3,8,-4,-2],
[3,7,-4,-1],[1,1,-1,1],[2,1,-1,1],[1,2,-1,1],[1,1,0,1],[1,1,-1,2],[3,4,-3,0],
[2,4,-2,0],[2,3,-2,1],[0,1,0,1],[1,1,0,1],[0,2,0,1],[0,1,1,1],[0,1,0,2],
[2,4,-2,0],[1,4,-1,0],[1,3,-1,1],[2,4,-3,0],[3,4,-3,0],[2,5,-3,0],[2,4,-2,0],
[2,4,-3,1],[4,7,-5,-1],[3,7,-4,-1],[3,6,-4,0],[0,3,-1,0],[1,3,-1,0],[0,4,-1,0],
[0,3,0,0],[0,3,-1,1],[2,6,-3,-1],[1,6,-2,-1],[1,5,-2,0],[0,2,-1,1],[1,2,-1,1],
[0,3,-1,1],[0,2,0,1],[0,2,-1,2],[2,5,-3,0],[1,5,-2,0],[1,4,-2,1],[1,5,-2,-1],
[2,5,-2,-1],[1,6,-2,-1],[1,5,-1,-1],[1,5,-2,0],[3,8,-4,-2],[2,8,-3,-2],
[2,7,-3,-1],[1,4,-2,0],[2,4,-2,0],[1,5,-2,0],[1,4,-1,0],[1,4,-2,1],[3,7,-4,-1],
[2,7,-3,-1],[2,6,-3,0]]
```

After repeatedly deleting an element that is a positive integer linear combination of others we are left with

$$[[-2,-3,2,2],[1,0,0,0],[-1,0,1,0],[0,0,-1,1],[1,3,-2,-1],[0,-1,0,1],[1,1,-1,0]]$$

Determinants of 4x4 minors are

$3,3,1,2,-3,1,-1,2,1,-1,0,1,2,1,2,0,-1,-2,1,1,3,-1,1,-2,-1,1,0,0,0,0,-1,-2,1,1,0$

If we instead take the starting matrix with $(1,2,-1,0)$ appended

$$[[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1],[2,3,-2,-1],[1,3,-1,-1],[1,2,-1,0]]$$

the 35-tuple is

$1,-1,-1,0,2,1,1,1,-1,-1,3,3,2,0,-2,-2,3,1,-1,2,-2,-1,-1,-1,1,1,0,0,0,0,3,1,-1,2,0$

which will agree when we take absolute value and sort by magnitude.

Their paper does a lot more, they handle ‘normalized Nash blowups’ and finite characteristic, and of course show not only that the 35 determinants match, but that it is because of a semigroup isomorphism coming from translating $\mathbb{Z}^4$ by $-(3,4,-1,-1)$ and applying a group isomorphism.

What it means

One thing this means is, phenomena in Math models of anything, in Physics or Biology, which have singularities, can't be assumed to need correction like fixing gimbal lock. It just isn't true that singular manifolds are nothing but incorrectly embedded natural smooth manifolds.

Actually, before I say that, I should mention, there are algebraic geometers at a very high level who consider the truth of such a notion in a deep way. Hironaka's work, subsequent to Nash and Semple, was very non-natural. Work of the mathematical experts goes into the Minimal Model theory and logarithmic theory which is very advanced, and already has notions about singularities which may exist naturally in some sense.

Stability

In view of this result, the thing to think about is what to do afterwards. It has to do with working *modulo* Nash transforms. Resolution questions then simplify because the cokernel of the map of differentials being zero can refer to a resolution which worked mod Nash transforms. To get this to work, it's necessary to consider sequences of Nash transforms in what they do to the algebra where they are known to be of finite length, not changing the geometry. When the geometry stops changing, Noetherianness really does mean the algebra stops changing after finitely many more steps. In this example, no matter how many times we take sums of six affinely general points, if we then take sums of six a.g. and then sums of five a.g., it is the first example where it will never be the same as taking first sums of five a.g. and then arbitrary sums of six. But for the equivalence, for the opposite implication to be true, we'd need to work modulo the operation of passing to sums of six affinely general. [1]

The main future implication is, except possibly in high echelons of Algebraic Geometry and category theory, no-one can say anymore that singularities aren't an intrinsic part of mathematical modelling. No one can say they just describe incorrect attempts at embeddings of smooth manifolds. Singularities are real things in Maths, and this means, whenever we use Maths to model anything in the real world, we can no longer blithely use manifold theory.

[1] see formula (1) of "Arc Lifting" where a module map being surjective is equivalent to an ideal resolving singularities *modulo* Nash transforms. Sequences of Nash transforms which stop changing the geometry still change the algebra, always just finitely many times. The main point is, it is algebraically possible to describe ideals whose blowups cause the Nash tower to be finite rather than those which are immediately smooth.

What is finite?

As I've tried to indicate in a footnote, there is something which is finite. One considers whether an ideal I is such that $Bl_I(V)$ *does* satisfy Nash's conjecture after k steps. The surjectivity which tests this can fail, but *the error in the number k is always finite*. Once k steps of Nash blowup of $Bl_I(V)$ become smooth, the algebra itself keeps changing with each new Nash step, but now Noetheriannesss *does* work, and it tells us that eventually this will be detected by surjectivity before blowing anything up. In the other direction, when surjectivity holds it means the cokernel from the pulled-back differentials to the differentials is actually zero, and *one* more Nash blowup suffices.

I'm not sure I ever thought we'd be here, but now one future direction really is to work modulo Nash transforms, and focus on the easier problem of finding I such that $V \mapsto Bl_I(V)$ changes V from one where Nash's conjecture fails to one where it succeeds.

That is, we can continue to believe Nash's intuition that Nash transforms are essentially identity maps, and what is new is that particular types of singularities really do exist even from that wider perspective.

Conclusion

Perhaps it is as significant a time as when Poincare and Einstein suggested that we shouldn't assume that dot products using Euclidean coordinates in the real line have meaning in physics. Even when people started defining abstract topological *spaces*, no one really thought that points of a set have to make sense independent of ordinary Galilean reference frames. Mathematics has always known that notation does not matter, and formalism does not matter. But great mathematicians, even like Nash, Abyahkhar, have to learn from thinking abstractly, and when the process debunks deep assumptions this is the best result. It is a natural experience of people wanting to be actually honest with each other.