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ON RESOLVING SINGULARITIES

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1. *Introduction*

Let V be an irreducible affine algebraic variety over a field k of characteristic zero, and let $(f_0, \ldots, f_m)$ be a sequence of elements of the coordinate ring. There is probably no elementary condition on the f_i and their derivatives that determines whether the blowup of V along $(f_0, \ldots, f_m)$ is nonsingular. The result of this paper is that there is indeed such an elementary condition, involving the first and second derivatives of the f_i, provided that we admit certain singular blowups, all of which can be resolved by an additional Nash blowup.

This paper is the promised sequel of [**3**], in which the same program was carried out for individual vector fields. Indeed, this paper generalizes the result of [**3**] to algebraic foliations of arbitrary codimension, and the case of codimension zero foliations corresponds to the problem of resolving the singularities of V.

Our results have a close connection with a question of Nash concerning resolutions. We now describe this briefly following Milnor [**2**], where further references may be found. Let $r = \dim(V)$. Suppose that $V = V_0 \subset W_0$ is an embedding in a nonsingular variety over k. Then V_0 lifts to a subvariety $V_1 \subset W_1 = \mathrm{Grass}_r(W_0)$ of the variety of r-planes in the tangent bundle of W_0. The natural map $\pi : V_1 \longrightarrow V_0$ is called the Nash blowup of V_0. It is the lowest blowup where $\pi^*(\Omega_{V_0/k})/\mathrm{torsion}$ is locally free. Now we can repeat the process, giving a variety $V_2 \subset W_2 = \mathrm{Grass}_r(W_1)$ and so on, and the question is that of whether eventually V_i is nonsingular.

There is a particular explicit sequence of ideals $R = J_0, J_1, J_2, \ldots \subset R$ so that $V_0 = \mathrm{Bl}_{J_0} V$, $V_1 = \mathrm{Bl}_{J_1} V$, $V_2 = \mathrm{Bl}_{J_2} V, \ldots$ with $J_i | J_{i+1}$ for all i. Applying the results of our earlier paper [**4**], we find that V_i is nonsingular if and only if the ideal class of J_{i+1} divides some power of the ideal class of J_i. This paper brings matters down to earth considerably: such a divisibility of ideal classes implies that, for this value of i and for some $N \geqslant r+2$,

$$J_i^{N-r-2} J_{i+1}^{r+3} = J_i^N J_{i+2}.$$

However, note that this identity in turn implies that J_{i+2} is a divisor of some power of J_{i+1}. Therefore, although V_i may fail to be nonsingular, when the identity holds, the *next* variety V_{i+1} must be nonsingular. Thus the Nash question is equivalent to the assertion that the identity above holds for some sufficiently large i and N.

2. *A toy theorem*

In order to explain the main theorem of this paper, Theorem 15, let us look first at what it says about an individual vector field δ on V when $k = \mathbb{C}$, V is an

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irreducible algebraic variety in $\mathbb{C}^3$, and $(f_0, \ldots, f_m)$ is the trivial (unit) ideal. Let R be the coordinate ring of V, and let K be the function field of V. Choose any nonzero vector field on V corresponding to a k linear derivation

$$\begin{aligned} \delta : R &\longrightarrow R \\ r &\longmapsto \dot{r}. \end{aligned}$$

Let (x, y, z) be the standard coordinates in $\mathbb{C}^3$, and let us define the velocity and acceleration vectors by

$$\begin{aligned} v &= (\dot{x}, \dot{y}, \dot{z}) \\ \dot{v} &= (\ddot{x}, \ddot{y}, \ddot{z}) \end{aligned}$$

Let (v) be the ideal generated by the entries of v, and let $(v \times \dot{v})$ be the ideal generated by the entries of the cross product. Let $S \subset V$ be the vanishing locus of $(\dot{x}, \dot{y}, \dot{z})$, and let $\tilde{V} \subset \mathbb{C}^3 \times \mathbb{P}^2$ be the closure of the graph of the familiar Gauss map

$$\begin{aligned} V - S &\longrightarrow \mathbb{P}^2 \\ (x, y, z) &\longmapsto [\dot{x} : \dot{y} : \dot{z}]. \end{aligned}$$

Note that if δ induces a nonsingular foliation on V, then it also induces a nonsingular foliation on $\tilde{V} = V$. Now, in this very special situation, we have the following theorem.

Toy Theorem. (i) *There is always an inclusion* $(v)^3 \subset (v)^3 + (v \times \dot{v})$.

(ii) *If δ induces a nonsingular foliation on V, then the inclusion above is an equality* $(v)^3 = (v)^3 + (v \times \dot{v})$.

(iii) *Conversely, if the inclusion is an equality, then δ induces a nonsingular foliation on $\tilde{V}$.*

Note that if V is a curve, then δ induces a nonsingular foliation on V (respectively $\tilde{V}$) if and only if V (respectively $\tilde{V}$) is nonsingular.

Theorem 15 is an analogous theorem that works not only for arbitrary affine varieties V, but also for an arbitrary blowup of V, and for foliations of any dimension. Moreover, the ideals that occur in the statement of Theorem 15 are all ideals of the original ring R.

The proof of the toy theorem works like this. Part (i) is obvious. Let $\pi : \tilde{V} \longrightarrow V$ be the natural map. For any sequence of elements $l = (l_0, \ldots, l_n)$ of K, we can consider the fractional ideal $(l) \subset K$, which is the set of R linear combinations of the l_i. For any such (l), we define a new fractional ideal $\mathscr{J}(l) = (l \times \dot{l}) + (l)^2(v)$ (where $(l \times \dot{l})$ is a fractional ideal with

$$\binom{n+1}{2}$$

entries).

Some properties of $\mathscr{J}(l)$ that were proven in [**3**] are as follows:

(1) $\pi^* \mathscr{J}(v) = \mathscr{O}_{\tilde{V}} \delta(\mathscr{O}_{\tilde{V}})(-2E)$;

(2) $(l)^2 \mathscr{J}(m) + (m)^2 \mathscr{J}(l) = \mathscr{J}(lm)$;

(3) $\mathscr{J}(1) = (v)$;

where π^* is the operation of pulling back ideals to ideal sheaves, lm is the product sequence, and E is the exceptional divisor associated to blowing up (v).

Here is the proof of (ii). Suppose that δ induces a nonsingular foliation on V.

This means that the fractional ideal (v) is invertible, that is, there is a sequence (w) of elements of K so that $(u)(w) = R$. Applying properties (2) and (3), we have

$$(w)^2(v \times \dot{v}) \subset (w)^2\mathscr{J}(v) \subset \mathscr{J}(vw) = \mathscr{J}(1) = (v).$$

Multiplying both sides by $(v)^2$ gives $(v \times \dot{v}) \subset (v)^3$, which proves the desired equality of ideals $\mathscr{J}(v) = (v)^3$.

Here is the proof of (iii). Suppose conversely that there is such an equality of ideals $\mathscr{J}(v) = (v)^3$. If π^* and 1 are applied, this yields

$$\mathscr{O}_{\tilde{V}}\delta(\mathscr{O}_{\tilde{V}})(-2E) \cong \mathscr{O}_{\tilde{V}}(-3E),$$

so $\mathscr{O}_{\tilde{V}}\delta(\mathscr{O}_{\tilde{V}}) \cong \mathscr{O}_{\tilde{V}}(-E)$ is locally free, which proves that δ induces a nonsingular foliation on $\tilde{V}$.

The main thing that the toy theorem is meant to illustrate is that the explicit and elementary condition that the inclusion $(v)^3 \subset (v)^3 + (v \times \dot{v})$ should be an equality is intermediate in strength between stating that δ induces a nonsingular foliation on V and stating that δ induces a nonsingular foliation on $\tilde{V}$.

Theorem 15 concerns our arbitrary irreducible affine variety V over our field k of characteristic zero with coordinate ring R, an arbitrary ideal $I = (f_0, \ldots, f_m) \subset R$, and an arbitrary algebraic foliation L on V (of any dimension). The theorem describes an elementary condition involving first and second derivatives that is intermediate in strength between stating that L lifts to a nonsingular foliation on $\mathrm{Bl}_I(V)$ and stating that L lifts to a nonsingular foliation on the Gauss blowup $\widetilde{\mathrm{Bl}_I(V)}$ of $\mathrm{Bl}_I(V)$ along L. An important aspect of Theorem 15 is that the ideals considered in the statement are all ideals in the original ring R.

Now I give the statement of Theorem 15. Let R be the coordinate ring of V, and let $x_1, \ldots, x_n$ be a sequence of k algebra generators of R. Recall that K is the fraction field of R. An algebraic (singular) foliation on V corresponds to a K sub Lie algebra $L \subset \mathrm{Der}_k(K, K)$ (see Section 3 below); let $\delta_1, \ldots, \delta_r$ be a K basis of L. We may assume that $\delta_i \in \mathrm{Der}_k(R, R)$. For any ideal $I = (f_0, \ldots, f_m)$ of R, let $\mathscr{J}(I)$ be the ideal generated by

$$f_{u_1} f_{u_2} \ldots f_{u_b} \cdot \det \begin{pmatrix} f_{i_1} & \delta_1 f_{i_1} & \ldots \delta_r f_{i_1} \\ & & \ldots \\ f_{i_a} & \delta_1 f_{i_a} & \ldots \delta_r f_{i_a} \\ 0 & \delta_1 x_{j_1} & \ldots \delta_r x_{j_1} \\ 0 & & \ldots \\ 0 & \delta_1 x_{j_b} & \ldots \delta_r x_{j_b} \end{pmatrix},$$

where a and b run over numbers such that $a + b = r + 1$, and where $0 \leqslant u_1, \ldots, u_b \leqslant m$, $0 \leqslant i_1 < i_2 < \ldots < i_a \leqslant m$, and $1 \leqslant j_1 < j_2 < \ldots < j_b \leqslant n$. The ideal $\mathscr{J}(I)$ is independent of the choice of generators $(f_0, \ldots, f_m)$.

For each choice of ideal I, the ideal $\mathscr{J}(I)$ has the property that the Gauss blowup $\widetilde{\mathrm{Bl}_I(V)}$ of $\mathrm{Bl}_I(V)$ along L satisfies $\widetilde{\mathrm{Bl}_I(V)} = \mathrm{Bl}_{I\mathscr{J}(I)}(V)$ (see Lemma 4).

For each choice of ideal I, the ideal $\mathscr{J}(I\mathscr{J}(I))$ makes sense, and it is generated by certain explicit expressions involving the f_i and the x_j and their first and second derivatives. Theorem 15 in this situation makes the following three assertions.

MAIN THEOREM. (i) *For any ideal I of R there is an inclusion $\mathscr{J}(I)^{r+2} \subset \mathscr{J}(I\mathscr{J}(I))$ of ideals of R.*

(ii) *If L lifts to a nonsingular foliation on* $\mathrm{Bl}_I(V)$*, then this inclusion becomes an* *equality after both sides are multiplied by a suitable Nth power of I:*

$$I^N \mathscr{J}(I)^{r+2} = I^N \mathscr{J}(I\mathscr{J}(I)).$$

(iii) *Conversely, if the equality in* (ii) *holds, then L does lift to a nonsingular foliation on the Gauss blowup $\widetilde{\mathrm{Bl}_I(V)}$ of $\mathrm{Bl}_I(V)$ along L.*

When L is the unique codimension zero foliation of V, then the lift of L to any blowup of V is just the unique codimension zero foliation of the blowup, which we may also call L. To say L is nonsingular on a blowup is the same as saying that the blowup is a nonsingular variety. Moreover, the Gauss blowup of $\mathrm{Bl}_I(V)$ along L is just the Nash blowup of $\mathrm{Bl}_I(V)$. Assembling these facts, we see that if I resolves the singularities of V, so $\mathrm{Bl}_I(V)$ is a nonsingular variety, then, for $r = \dim(V)$ and some N, the inclusion $I^N\mathscr{J}(I)^{r+2} \subset I^N\mathscr{J}(I\mathscr{J}(I))$ becomes an equality, and conversely, when this is so, the Nash blowup $\widetilde{\mathrm{Bl}_I(V)}$ is a nonsingular variety.

3. *Singular foliations*

Let V be an affine irreducible variety over a field k. V is determined by its coordinate ring, an arbitrary finite type k-algebra R without zero divisors. Let K be the fraction field of R. Let us say that a *singular foliation* on V is just any K linear Lie sub algebra $L \subset \mathrm{Der}_k(K,K)$. We shall let $\hat{L} = \mathrm{Hom}_K(L,K)$, and we shall let $\overline{\Omega_{V/k}}$ be the image of the homomorphism

$$\Omega_{V/k} \longrightarrow \hat{L}$$

that sends a differential of the form dx to the functional $(\delta \mapsto \delta(x))$. Let us record this formula to avoid any confusion:

$$\overline{\Omega_{V/k}} = \mathrm{Image}(\Omega_{V/k} \longrightarrow \hat{L}).$$

We shall say that the singular foliation is *nonsingular on V* if this image R-module is projective. The following proposition justifies this definition.

Proposition 1. *If L is nonsingular on V, then there are elements $f_i \in R$ generating the unit ideal such that, for each i, each Lie ring $L \cap \mathrm{Der}_k(R[f_i^{-1}], R[f_i^{-1}])$ is free over $R[f_i^{-1}]$ with a basis $\delta_1, \dots, \delta_r$ (depending on i) such that there are elements $x_1, \dots, x_r \in R[f^{-1}]$ with $\delta_i(x_j) = 1$, $i = j$, and $\delta_i(x_j) = 0$, $i \neq j$.*

Proof. Apply $\mathrm{Hom}_R(-,R) \subset \mathrm{Hom}_R(-,K)$ to the split surjection $\Omega_{V/k} \longrightarrow \overline{\Omega_{V/k}}$ to obtain a pullback square showing

$$L \cap \mathrm{Der}_k(R,R) = \mathrm{Hom}_R(\overline{\Omega_{V/k}}, R).$$

We may assume that $\overline{\Omega_{V/k}}$ is free with basis of the form $dx_1, \dots, dx_r$, and let $\delta_1, \dots, \delta_r$ be the dual basis. □

Even if V is a non-affine irreducible variety with function field K, for L a K-linear Lie algebra $L \subset \mathrm{Der}_k(K,K)$, we may still think of L as defining a singular foliation on V. L is said to be nonsingular just if it is nonsingular on each affine part, and the proposition above still holds for each affine part of V.

If W is any variety birationally equivalent to V, and if $L \subset \mathrm{Der}_k(K,K)$ defines a singular foliation on V, then note that L also defines a singular foliation on W because V and W share the same function field. In particular, if W is a blowup of V, then we may consider the question of whether the singular foliation L on V becomes nonsingular on W.

In what follows, we will restrict ourselves to the case in which V is affine with coordinate ring R, furnished with a singular foliation L, and we will study ideals $I \subset R$ to answer the question of whether there exists an ideal I such that L becomes nonsingular on $\tilde{V} = \mathrm{Bl}_I(V)$.

4. *The module M_γ*

The previous section leads to the question of how to compute $\overline{\Omega_{\tilde{V}/k}}$ in terms of $\overline{\Omega_{V/k}}$ and the ideal I. In these terms there is a natural answer: there is a class depending on I

$$\gamma \in \mathrm{Ext}^1_R(I, I \otimes \Omega_{V/k})$$

that has a natural image, which we will also call γ, in each $\mathrm{Ext}^1_R(I, I\overline{\Omega_{V/k}})$, and this class defines a certain R-module M_γ with a structure map $p : M_\gamma \longrightarrow I$. (The referee of this paper noted that the class γ is the well known Atiyah class (see [**2**, Section 4]), and therefore the module M_γ is equal to an appropriate quotient of the module of principal parts of I. In other words, M_γ is an appropriate quotient of the module of global 1-jets of sections of I viewed as a coherent sheaf.) In Proposition 2, we will show that there is an exact sequence of sheaves on $\tilde{V} = \mathrm{Bl}_I(V)$

$$0 \longrightarrow \overline{\Omega_{\tilde{V}/k}} \longrightarrow \overline{\pi^*(M_\gamma)}(E) \xrightarrow{\overline{\pi^* p(E)}} \mathcal{O}_{\tilde{V}} \longrightarrow 0, \tag{1}$$

where E is the exceptional divisor. Here the overline in the middle term refers to reduction modulo torsion. $\pi : \tilde{V} \longrightarrow V$ is the structure map of the blowup. The map $\overline{\pi^* p(E)}$ is the result of pulling back via π the map $p : M_\gamma \longrightarrow I$, twisting by E, and reducing modulo torsion.

Now is a suitable time to explain the convention that we will use throughout this paper. An overline on a module or coherent sheaf will always denote the torsion-free quotient of that module or sheaf, except that, for any variety V over k, the symbol $\overline{\Omega_{V/k}}$ will denote the natural image of $\Omega_{V/k}$ in $\hat{L}$, which we will call the *reduced* differentials, and, more generally, an overline over $\wedge^r\Omega_{V/k}$ will denote the natural image in $\wedge^r\hat{L}$. Let us also state the more general formula to avoid any confusion:

$$\overline{\wedge^r\Omega_{V/k}} = \mathrm{Image}(\wedge^r\Omega_{V/k} \longrightarrow \wedge^r\hat{L}).$$

Finally, when V is not affine, we have a similar definition for twisted sheaves that will play a role in Section 5. When L is the whole of $\mathrm{Der}_k(K, K)$, these notions coincide, the torsion-free quotient being the image in $\hat{L}$.

The class γ is defined as follows. Take a resolution of I

$$R^a \longrightarrow R^b \longrightarrow R^c \xrightarrow{\epsilon} I \longrightarrow 0,$$

and say that the middle map is

$$g_i \longmapsto \sum_j a_{ij} h_j,$$

where g_i and h_j are respective basis vectors. Then a cocycle $z : R^b \longrightarrow I\overline{\Omega_{V/k}}$

$$\begin{array}{ccccccccc}
R^a & \longrightarrow & R^b & \longrightarrow & R^c & \longrightarrow & I & \longrightarrow & 0 \\
\downarrow & & \downarrow z & & \downarrow e & & \downarrow & & \\
0 & \longrightarrow & I\overline{\Omega_{V/k}} & \subset & \overline{\Omega_{V/k}} & \longrightarrow & 0 & &
\end{array}$$

FIGURE 1.

representing γ sends g_i to $-\sum_j \epsilon(h_j)\bar{d}a_{ij}$, where $\bar{d}$ is the natural derivation $R \longrightarrow \overline{\Omega_{V/k}}$. The Leibniz rule implies that this map satisfies the cocycle condition. We can build the extension module M_γ using the double complex shown in Figure 1, where the vertical map e is given by

$$e(h_j) = \bar{d}\epsilon h_j.$$

It is easy to see that the diagram in Figure 1 commutes. A copy of M_γ occurs as the submodule of $I \oplus \overline{\Omega_{V/k}}$ generated by the image of $I\overline{\Omega_{V/k}} + R^c$. This in turn is equal to the submodule of $I \oplus \hat{L}$ generated by the $f \oplus \bar{d}f$ for $f \in I$. The projection $I \oplus \hat{L} \longrightarrow I$ induces a surjection $M_\gamma \longrightarrow I$, and the kernel is exactly $I\overline{\Omega_{V/k}}$. One caution is that if $(f_0, \ldots, f_n)$ is a sequence of generators of I, then it does not automatically follow that the rows $(f_i, \bar{d}f_i)$ generate M_γ. It is only the case that these rows together with a system of generators of $I\overline{\Omega_{V/k}}$ suffice.

PROPOSITION 2. *Sequence* (1) *is exact; that is, the kernel of* $\overline{\pi^* p(E)}$ *is the image in* $\hat{L}$ *of the sheaf of differentials of* $\tilde{V}$.

Proof. Choose once and for all a K basis $\delta_1, \ldots, \delta_r$ of L so we have $\hat{L} \cong K^r$ by which each element v is sent to $(v(\delta_1), \ldots, v(\delta_r))$. The extension module M_γ is then isomorphic to the submodule of $I \oplus K^r$ generated by all rows $(f, \delta_1 f, \ldots, \delta_r f)$ for $f \in I$. Because of the statement in the sentence preceding Proposition 2, if I is generated by $f_0, \ldots, f_n$, we can view M_γ as the module of $I \oplus K^r$ generated by rows of the following two types:

$$\begin{pmatrix} f_i & \delta_i(f_i) & \ldots & \delta_r(f_i) \\ 0 & f_j\delta_1(x_c) & \ldots & f_j\delta_r(x_c) \end{pmatrix}.$$

The sheaf $\overline{\pi^* M_\gamma}(E)$ can be constructed chart by chart. The 0th coordinate chart of $\tilde{V}$ is $U = \operatorname{Spec}(\tilde{R})$ for $\tilde{R} = R[f_1/f_0, \ldots, f_n/f_0]$, and the module $\overline{\pi^* M_\gamma}(E)(U)$ over this ring can be explicitly constructed within $K \oplus K^r$ by multiplying each row above by f_0^{-1} and considering the $\tilde{R}$ module that the new rows generate. Some typical rows that result are

$$\begin{pmatrix} 1 & f_0^{-1}\delta_1(f_0) & \ldots & f_0^{-1}\delta_r(f_0) \\ f_i/f_0 & f_0^{-1}\delta_1(f_i) & \ldots & f_0^{-1}\delta_r(f_i) \\ 0 & \delta_1(x_c) & \ldots & \delta_r(x_c) \\ 0 & f_j/f_0\delta_1(x_c) & \ldots & f_j/f_0\delta_r(x_c) \end{pmatrix}.$$

Subtracting f_i/f_0 times the first row from the second and f_j/f_0 times the third row from fourth yields

$$\begin{pmatrix} 1 & f_0^{-1}\delta_1(f_0) & \dots & f_0^{-1}\delta_r(f_0) \\ 0 & f_0^{-1}\delta_1(f_i) - f_i/f_0^2\delta_1(f_0) & \dots & f_0^{-1}\delta_r(f_i) - f_i/f_0^2\delta_r(f_0) \\ 0 & \delta_1(x_c) & \dots & \delta_r(x_c) \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

The kernel of the projection on $\tilde{R} \oplus 0$ is generated by rows such as the second and third above, and using the rule for differentiating a quotient, we see these are just the images of the k algebra generators x_c and f_i/f_0 of $\tilde{R}$. The $\tilde{R}$ module that they span is the image of the natural map

$$\Omega_{\tilde{R}/k} \longrightarrow \hat{L},$$

which is $\overline{\Omega_{\tilde{V}/k}}(U)$ as claimed. The same considerations apply to each other coordinate chart, and this proves that the kernel of the projection $\overline{\pi^* M_\gamma}(E) \longrightarrow \mathcal{O}_{\tilde{V}}$ is $\overline{\Omega_{\tilde{V}/k}}$. $\square$

It follows from Proposition 2 that the reduced differentials of $\tilde{V}$ are locally free if and only if the pullback of M_γ modulo torsion is locally free, so the question of resolving the singular foliation L on V comes down to finding an I such that the associated extension module M_γ pulls back to a locally free sheaf modulo torsion.

There is a lowest blowup τ that makes $\overline{\tau^* M_\gamma}$ projective, namely the blowup of the rank torsion-free module $\overline{\wedge^{r+1} M_\gamma}$. We can make a fractional ideal $\mathscr{J}(I)$ isomorphic to this module, namely the fractional ideal generated by the determinants of all possible matrices

$$\begin{pmatrix} f_0 & \delta_1 f_0 & \dots & \delta_r f_0 \\ & & \dots & \\ f_r & \delta_1 f_r & \dots & \delta_r f_r \end{pmatrix},$$

where the δ_i are our fixed basis of L, and $f_0, \dots, f_r$ ranges over all possible lists of $r+1$ elements of I. If the δ_i are chosen to belong to $L \cap \mathrm{Der}_k(R,R)$, then $\mathscr{J}(I)$ will be an ordinary ideal instead of a fractional ideal, but this is an unimportant limitation because we will want to apply the operator $\mathscr{J}$ to fractional ideals anyway.

Note that, because of Proposition 2 (the exactness of sequence (1)), we have the following.

COROLLARY 3.

$$\overline{\pi^* \mathscr{J}(I)} \cong \overline{\pi^* \wedge^{r+1} M_\gamma} \cong \overline{\wedge^r \Omega_{\tilde{V}/k}(-E))} \otimes \mathcal{O}_{\tilde{V}}(-E) \cong \overline{\wedge^r \Omega_{\tilde{V}/k}}(-E - rE).$$

Because twisting does not affect blowing up, the blowup of $\tilde{V} = \mathrm{Bl}_I(V)$ along $\overline{\pi^* \mathscr{J}(I)}$ is the same as the blowup of $\tilde{V}$ along $\overline{\wedge^r \Omega_{\tilde{V}/k}}$. This is in turn isomorphic as a variety over V to the Gauss blowup of $\mathrm{Bl}_I(V)$ along L; let us call this $\widetilde{\mathrm{Bl}_I(V)}$. Thus we have the following isomorphisms of varieties over V:

$$\begin{aligned} \widetilde{\mathrm{Bl}_I(V)} &= \mathrm{Bl}_{\overline{\wedge^r \Omega_{\mathrm{Bl}_I(V)/k}}} \mathrm{Bl}_I(V) \\ &\cong \mathrm{Bl}_{\overline{\wedge^r \Omega_{\mathrm{Bl}_I(V)/k}(-E-rE)}} \mathrm{Bl}_I(V) \\ &\cong \mathrm{Bl}_{\overline{\pi^* \mathscr{J}(I)}} \mathrm{Bl}_I(V) \\ &\cong \mathrm{Bl}_{I \mathscr{J}(I)} V. \end{aligned}$$

Let us record this as a little lemma.

LEMMA 4. *The Gauss blowup of* $\mathrm{Bl}_I(V)$ *along* L *is isomorphic as a variety over* V *to* $\mathrm{Bl}_{I\mathscr{J}(I)}(V)$.

Recall that $x_1, \ldots, x_n$ are a system of k-algebra generators of R.

PROPOSITION 5. *If* I *is generated by* $f_0, \ldots, f_m$, *then* $\mathscr{J}(I)$ *is generated by the elements*

$$f_{u_1} f_{u_2} \ldots f_{u_b} \cdot \det \begin{pmatrix} f_{i_1} & \delta_1 f_{i_1} & \ldots & \delta_r f_{i_1} \\ & & \ldots & \\ f_{i_a} & \delta_1 f_{i_a} & \ldots & \delta_r f_{i_a} \\ 0 & \delta_1 x_{j_1} & \ldots & \delta_r x_{j_1} \\ 0 & & \ldots & \\ 0 & \delta_1 x_{j_b} & \ldots & \delta_r x_{j_b} \end{pmatrix},$$

where a *and* b *run over numbers such that* $a+b = r+1$, *and where* $0 \leqslant u_1, \ldots, u_b \leqslant m$, $0 \leqslant i_1 < i_2 < \ldots < i_a \leqslant m$ *and* $1 \leqslant j_1 < j_2 < \ldots < j_b \leqslant n$.

Proof. In the first displayed expression in the proof of Proposition 2, we saw that the image of M_γ is generated by the rows

$$(f_i \quad \delta_1(f_i) \quad \ldots \quad \delta_r(f_i))$$

and the rows

$$(0 \quad f_j \delta_1(x_c) \quad \ldots \quad f_j \delta_r(x_c)).$$

The size $r+1$ square matrix above is obtained by choosing $r+1$ such rows in all possible ways and taking determinants. Therefore the determinants generate the corresponding image of $\overline{\wedge^{r+1} M_\gamma}$ in K. $\square$

The problem of resolving the singular foliation L on V comes down to finding an ideal $I \subset R$ such that $\overline{\wedge^r \Omega_{\mathrm{Bl}_I(V)/k}}(-E - rE) \cong \overline{\pi^* \mathscr{J}(I)}$ is locally free. This happens if and only if $\mathrm{Bl}_I(V)$ dominates $\mathrm{Bl}_{\mathscr{J}(I)}(V)$. Luckily we have the following theorem.

THEOREM 6 [**4**]. *Let* $I, J \subset R$ *be ideals. Then* $\mathrm{Bl}_I(V)$ *dominates* $\mathrm{Bl}_J(V)$ *if and only if there is a number* α *and a fractional ideal* S *such that*

$$JS = I^\alpha.$$

Although the theorem is stated for ideals, it follows for fractional ideals. Thus we have the following corollary.

COROLLARY 7. *An ideal* $I \subset R$ *has the property that* L *lifts to a nonsingular foliation on* $V = \mathrm{Bl}_I(V)$ *if and only if* $\mathscr{J}(I)$ *is a divisor of a power of* I *as a fractional ideal, that is, if and only if there is a fractional ideal* S *of* R *and a number* α *such that*

$$S\mathscr{J}(I) = I^\alpha.$$

Proof. This is just a matter of assembling data already proven. By definition, L lifts to a nonsingular foliation on $\tilde{V} = \mathrm{Bl}_I(V)$ if and only if $\overline{\Omega_{\tilde{V}/k}}$ is locally free. This happens if and only if $\overline{\wedge^r \Omega_{\tilde{V}/k}}$ is locally free. Also, we have from Corollary 3 that $\overline{\wedge^r \Omega_{\tilde{V}/k}} \cong \overline{\pi^* \mathscr{J}(I)}(E + rE)$. This is locally free if and only if the blowup $\mathrm{Bl}_I(V)$ dominates $\mathrm{Bl}_{\mathscr{J}(I)}(V)$, and by Theorem 6 this happens if and only if there is a fractional ideal S of R and a number α so that $\mathscr{J}(I)S = I^\alpha$. $\square$

If we take the case $r = 1$ as a guide [**3**], we should not expect to have a formula that will simply give us the generating sequence $(f_0, \dots, f_m)$ for an ideal that will resolve the singularities of V. However, we may hope to have an elementary condition on the f_i and their derivatives that will tell us whether the ideal or an associated ideal will resolve them.

5. *Calculation of the reduced differentials of the blowup*

The idea of this section, which is independent of the rest of the paper, is to describe the differentials $\Omega_{\tilde{V}/k}$, of a blowup $\pi : \tilde{V} \longrightarrow V$, or rather the image $\overline{\Omega_{\tilde{V}/k}} \subset \hat{L}$. When $L = \mathrm{Der}_k(K, K)$, this is just the torsion-free quotient of the differentials of $\tilde{V}$. In the following section we will return to the problem of determining nonsingularity of the foliation lifted to the blowup solely in terms of the generators of the ideal downstairs. In this section, though, we will allow ourselves to work with sheaves upstairs in the blowup. Throughout this section, V will be affine and irreducible over a field k, with coordinate ring R.

The first step is to notice that, for any ideal $I \subset R$ letting $\Omega = \Omega_{V/k}$, the canonical derivation $d : R \longrightarrow \Omega_{V/k}$ defines a *homomorphism* of modules

$$h : I \longrightarrow \Omega/I\Omega,$$

because for $f \in I$ and $r \in R$, we have $d(rf) = rdf + fdr$, and the second term is in $I\Omega$.

Let $\tilde{V} = \mathrm{Bl}_I(V)$; then there are natural inclusions of sheaves

$$\overline{\pi^*\Omega}(-E) \subset \overline{\Omega_{\tilde{V}/k}}(-E) \subset \overline{\pi^*\Omega}$$

that define $\overline{\Omega_{\tilde{V}/k}}(-E)$ as the inverse image in $\overline{\pi^*\Omega}$ of a certain subsheaf $\mathscr{L} \subset \overline{\pi^*\Omega}/\overline{\pi^*\Omega}(-E)$.

We shall describe the sheaf $\mathscr{L}$.

THEOREM 8. *The desired subsheaf $\mathscr{L}$ is the image of the composite*

$$(\pi^*(I))_{\mathrm{tors}} \subset \pi^*(I) \xrightarrow{\pi^*h} \pi^*(\Omega/I\Omega) \longrightarrow \overline{\pi^*\Omega}/\overline{\pi^*\Omega}(-E).$$

Proof. Recall that there is a certain R-module M_γ defined earlier that fits into the pullback diagram shown in Figure 2 and defines an isomorphism $\bar{\Omega}/I\bar{\Omega} \longrightarrow (I \oplus \bar{\Omega})/M_\gamma$.

The diagram in Figure 2 does not of course remain a pullback after pulling back by π, but the important property of M_γ is that we do obtain a similar diagram (see Figure 3) by applying π^* just to the maps $M_\gamma \longrightarrow I$ and $I \oplus \bar{\Omega} \longrightarrow I$, reducing mod torsion and taking kernels.

This gives us an isomorphism $\overline{\pi^*\Omega}/\overline{\Omega_{\tilde{V}/k}(-E)} \longrightarrow (\mathscr{O}_{\tilde{V}}(-E) \oplus \overline{\pi^*\Omega})/(\overline{\pi^* M_\gamma})$. We now compare the middle row of the diagram in Figure 3 with the result of pulling back the middle row of the diagram in Figure 2, but making substitutions according to the two isomorphisms we have so far discovered. We obtain the diagram with exact rows and surjective vertical maps shown in Figure 4.

The sequence of the first two kernels splices to the sequence of two displayed cokernels to give the four term exact sequence

$$(\pi^* M_\gamma)_{\mathrm{tors}} \longrightarrow (\pi^* I)_{\mathrm{tors}} \oplus (\pi^*\bar{\Omega})_{\mathrm{tors}} \longrightarrow \pi^*(\bar{\Omega}/I\bar{\Omega}) \longrightarrow \overline{\pi^*\Omega}/\overline{\Omega_{\tilde{V}/k}(-E)} \longrightarrow 0.$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & I\bar{\Omega} \arrow[r] \arrow[d] & \bar{\Omega} \arrow[r] \arrow[d] & \bar{\Omega}/I\bar{\Omega} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_\gamma \arrow[r] \arrow[d] & I \oplus \bar{\Omega} \arrow[r] \arrow[d] & \dfrac{(I \oplus \bar{\Omega})}{M_\gamma} \arrow[r] & 0 \\
 & I \arrow[r, phantom, "="] \arrow[d] & I \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

FIGURE 2.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \overline{\Omega_{\tilde{V}/k}(-E)} \arrow[r] \arrow[d] & \overline{\pi^*\Omega} \arrow[r] \arrow[d] & \overline{\pi^*\Omega}/\overline{\Omega_{\tilde{V}/k}(-E)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{\pi^* M_\gamma} \arrow[r] \arrow[d] & \mathcal{O}_{\tilde{V}}(-E) \oplus \overline{\pi^*\Omega} \arrow[r] \arrow[d] & (\mathcal{O}_{\tilde{V}}(-E) \oplus \overline{\pi^*\Omega})(\overline{\pi^* M_\gamma}) \arrow[r] & 0 \\
 & \mathcal{O}_{\tilde{V}}(-E) \arrow[r, phantom, "="] \arrow[d] & \mathcal{O}_{\tilde{V}}(-E) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

FIGURE 3.

$$
\begin{tikzcd}
 & \pi^* M_\gamma \arrow[r] \arrow[d] & \pi^* I \oplus \pi^*\Omega \arrow[r] \arrow[d] & \pi^*(\bar{\Omega}/I\bar{\Omega}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{\pi^* M_\gamma} \arrow[r] & \mathcal{O}_{\tilde{V}}(-E) \oplus \overline{\pi^*\Omega} \arrow[r] & \overline{\pi^*\Omega}/\overline{\Omega_{\tilde{V}/k}(-E)} \arrow[r] & 0
\end{tikzcd}
$$

FIGURE 4.

$$\begin{tikzcd}
\overline{\Omega_{\tilde{V}/k}(-E)} \arrow[r] \arrow[d] & \pi^*\overline{\Omega} \arrow[d] \\
\mathscr{L} \arrow[r, phantom, "\subset"] & \overline{\pi^*\Omega}/\overline{\pi^*\Omega(-E)}
\end{tikzcd}$$

FIGURE 5.

If we take into account the fact that

$$\frac{\pi^*(\bar{\Omega}/I\bar{\Omega})}{\mathrm{Image}((\pi^*\bar{\Omega})_{\mathrm{tors}})} \cong \overline{\pi^*\Omega}/\overline{\pi^*\Omega(-E)},$$

this gives us the exact sequence

$$(\pi^* I)_{\mathrm{tors}} \longrightarrow \overline{\pi^*\Omega}/\overline{\pi^*\Omega(-E)} \longrightarrow \overline{\pi^*\Omega}/\overline{\Omega_{\tilde{V}/k}(-E)} \longrightarrow 0.$$

Here the subsheaf described explicitly in the statement of the theorem is the image of the leftmost map, and the sheaf $\mathscr{L}$ which determines the differentials of $\tilde{V}$ is the kernel of the second map. The fact that they are equal therefore follows by exactness. □

COROLLARY 9. *We can reconstruct $\overline{\Omega_{\tilde{V}/k}(-E)}$ as the pullback shown in Figure* 5, *where $\mathscr{L}$ is the sheaf explicitly described in the statement of Theorem* 8.

6. *Study of $\mathscr{J}$*

We shall study the effect of $\mathscr{J}$ on powers I^n. We first have a lemma that, although not necessary for the main result, has nevertheless led to a simplification. I wish to thank D. Rumynin for pointing out this proposition.

PROPOSITION 10. *Let R be a k algebra of finite type, k be a field of characteristic zero, and I be an ideal of R. Let $r \geqslant 0$. Then there is a system of generators $(f_0, \ldots, f_m)$ of I such that $(f_0^{r+1}, \ldots, f_m^{r+1}) = I^{r+1}$.*

Proof. Start with a sequence $(f_0, \ldots, f_{m'})$ that generates I. Then I^{r+1} is generated by all degree $r+1$ monomials in the f_i. Each such monomial $f_0^{i_0} f_1^{i_1} \ldots f_{m'}^{i_{m'}}$ is equal to $1/(r+1)!$ times the alternating sum of the $(r+1)$th powers of the subsums of the expression

$$\underbrace{f_0 + \ldots + f_0}_{i_0 \text{ times}} + \ldots + \underbrace{f_{m'} + \ldots + f_{m'}}_{i_{m'} \text{ times}}.$$

The additional generators can be taken to be the subsums. □

At this stage, it turns out to be a good idea to define, for any sequence of elements $(f_1, \ldots, f_m)$ in I, the 'wrong' fractional ideal $\mathscr{M}(f_1, \ldots, f_m)$ to be generated by the determinants resulting from these generators only:

$$\det \begin{pmatrix} f_{i_0} & \delta_1 f_{i_0} & \ldots & \delta_r f_{i_0} \\ & & \ldots & \\ f_{i_r} & \delta_1 f_{i_r} & \ldots & \delta_r f_{i_r} \end{pmatrix}.$$

This is contained in $\mathscr{J}(I)$, but the inclusion is proper in general. However, if the generating sequence $(f_0,\dots,f_m)$ is appropriately enlarged, then the inclusion becomes an equality.

PROPOSITION 11. *If we begin with a sequence of generators $f_0,\dots,f_m$ of a (fractional) ideal I of R, and extend by appending all products with a system of k algebra generators of R, the new sequence $(f_0,\dots,f_{m'})$ has the property that $\mathscr{J}(I) = \mathscr{M}(f_1,\dots,f_{m'})$.*

Proof. It suffices to show that the rows $(f_i \;\; \delta_1(f_i)\dots\delta_r(f_i))$ generate M_γ. We know that M_γ is generated by rows of the type above together with rows $(0 \;\; f_i\delta_1(x_j)\dots f_i\delta_r(x_j))$, both with $i \leqslant m$. Rows of the first type belong to our proposed generating set. To obtain rows of the second type, choose s such that $f_s = x_j f_i$. Then

$$\begin{aligned}
&(f_s \;\; \delta_1(f_s)\dots\delta_r(f_s)) - x_j(f_i \;\; \delta_1(f_i)\dots\delta_r(f_i))\\
&\quad = (0 \;\; \delta_1(x_jf_i) - x_j\;\delta_1(f_i)\dots\delta_r(x_jf_i) - x_j\;\delta_r(f_i))\\
&\quad = (0 \;\; f_i\delta_1(x_j)\dots f_i\delta_r(x_j)),
\end{aligned}$$

as needed. □

Suppose that V is an irreducible variety over a field k of characteristic zero, R is the coordinate ring of V, and K is the function field of V. Let $L \subset \mathrm{Der}_k(K,K)$ be a singular algebraic foliation on V. Fix $\delta_1,\dots,\delta_r$ as a K-basis of L.

THEOREM 12. *Let I and J be fractional ideals of R.*

$$I^{(r+1)}\mathscr{J}(J) \subset \mathscr{J}(IJ).$$

Proof. Choose a sequence of generators of each ideal. Since the characteristic of k is zero, we can include enough generators f_i of I so that the powers f_i^{r+1} generate I^{r+1}. Extend both generating sequences by appending all multiples of the generators with the k-algebra generators x_i of R. Call the new sequences $(f_0,\dots,f_m)$ and $(g_0,\dots,g_u)$. Note that the product sequence (f_sg_t) contains a system of generators of IJ as well as all products of these generators with the x_i. Therefore, by Proposition 11, we can write

$$\begin{aligned}
\mathscr{M}(f_0,\dots,f_m) &= \mathscr{J}(I)\\
\mathscr{M}(g_0,\dots,g_u) &= \mathscr{J}(J)\\
\mathscr{M}((f_0,\dots,f_m)(g_0,\dots,g_u)) &= \mathscr{J}(IJ).
\end{aligned}$$

By our choice of generators of I, we also have by Proposition 10 (suitably adapted for fractional ideals)

$$(f_0^{r+1},\dots,f_m^{r+1}) = I^{r+1}.$$

A typical generator of $\mathscr{M}((f_0,\dots,f_m)(g_0,\dots,g_u))$ is a determinant of a size $r+1$ matrix with rows that look like

$$(f_sg_t, f_s\delta_1g_t + g_t\delta_1f_s,\dots,f_s\delta_rg_t + g_t\delta_rf_s)$$

for various choices of s and t. Each column after the first of such a matrix is a sum of two columns in an obvious way. The determinant is therefore a sum of the 2^r

determinants where we have chosen either the first or second column in each case. If we choose the same value of s for each row, all these determinants vanish except for one, which is the determinant of a matrix with rows that look like

$$(f_s g_t, f_s \delta_1 g_t, \ldots, f_s \delta_r g_t)$$

for various values of t. All the other matrices in the sum have at least one column that is a multiple of the first column by an element of K, so their determinants vanish. The determinant of the one matrix that counts is f_s^{r+1}, times an arbitrary generator of $\mathscr{M}(g_0, \ldots, g_u)$. Repeating the calculation for each value of s gives

$$(f_0^{r+1}, \ldots, f_m^{r+1})\mathscr{M}(g_0, \ldots, g_u) \subset \mathscr{M}((f_0, \ldots, f_m)(g_0, \ldots, g_u)).$$

Combining facts, we have

$$\begin{aligned} I^{r+1}\mathscr{J}(J) = (f_0^{r+1}, \ldots, f_m^{r+1})\mathscr{J}(J) &= (f_0^{r+1}, \ldots, f_m^{r+1})\mathscr{M}(g_0, \ldots, g_u) \\ &\subset \mathscr{M}((f_0, \ldots, f_m)(g_0, \ldots, g_u)) = \mathscr{J}(IJ). \end{aligned}$$

□

Applying this result plus induction, we have

$$I^{(r+1)(N-1)}\mathscr{J}(I) \subset \mathscr{J}(I^N).$$

We have thus bounded $\mathscr{J}(I^N)$ from below. What is remarkable is that we can bound it from above, and the bounds will be equal, so we will have calculated $\mathscr{J}(I^N)$. Both bounding arguments will have used the fact that the characteristic of k is zero, for two different reasons.

LEMMA 13. *Let I be any fractional ideal of R for R as above. Then $\mathscr{J}(I^N) \subset I^{(N-1)(r+1)}\mathscr{J}(I)$.*

Proof. Let $f_0, \ldots, f_m$ be a generating sequence of I chosen by Proposition 11 so that $\mathscr{M}(f_0, \ldots, f_m) = \mathscr{J}(I)$. Note that the sequence of degree N monomials in the f_i becomes extended at the same time in the appropriate way to satisfy the hypothesis of Proposition 11 so that $\mathscr{J}(I^N)$ is equal to $\mathscr{M}$ applied to the sequence of degree N monomials in the f_i. The latter is generated by the determinants of certain matrices. Let us now look at the case $N = 3$, the general case being similar. Each row of the typical matrix looks like $(f_i f_j f_k,\ \delta_1(f_i f_j f_k), \ldots, \delta_r(f_j f_j f_k))$. Expanding out using the Leibniz rule, one obtains a sum of three rows, namely

$$f_i f_j (f_k/3,\ \delta_1(f_k), \ldots, \delta_r(f_k))$$

$$f_i f_k (f_j/3,\ \delta_1(f_j), \ldots, \delta_r(f_j))$$

and

$$f_j f_k (f_i/3,\ \delta_1(f_i), \ldots, \delta_r(f_i)).$$

Because of the multilinearity of the determinant, our expression for the determinant is $1/3$ times a sum of degree $2(r+1)$ monomials in the f_i times determinants of the matrices that come into the definition of $\mathscr{M}(f_0, \ldots, f_m)$. Thus each term is an element of $I^{2(r+1)}\mathscr{M}(f_0, \ldots, f_m) = I^{2(r+1)}\mathscr{J}(I)$ as needed. The proof clearly generalizes to arbitrary N. □

By some miracle, the lemmas above are precise converses of each other, so we get an equality of ideals, at least when the characteristic of k is zero.

THEOREM 14. *Suppose that* $\mathrm{char}(k) = 0$. *Let* I *be a fractional of* R, *and let* r *be the dimension of* L *over* K. *Then*

$$\mathscr{J}(I^N) = I^{(N-1)(r+1)}\mathscr{J}(I).$$

We will use Theorem 12 and Theorem 14 in the proof of Theorem 15. There it will again happen that separate arguments will furnish upper and lower bounds for an ideal, and these will match exactly.

7. *The main theorem*

Let R be an integral domain that is a k algebra of finite type for k a field of characteristic zero. Let $V = \mathrm{Spec}(R)$. Let $L \subset \mathrm{Der}_k(K,K)$ be a K-linear sub Lie algebra, and let $\delta_1,\dots,\delta_r$ be a basis of L.

For a fractional ideal J of R, recall that $\mathscr{J}(J)$ is the fractional ideal of R generated by the determinants

$$\det\begin{pmatrix} f_0 & \delta_1 f_0 & \dots & \delta_r f_0 \\ & & \dots & \\ f_r & \delta_1 f_r & \dots & \delta_r f_r \end{pmatrix}$$

for $f_0,\dots,f_r \in J$. An explicit finite list of generators of $\mathscr{J}(J)$ is given in Proposition 5 if $f_0,\dots,f_m$ generate J. We can arrange that $\mathscr{J}(J)$ is an ordinary ideal instead of a fractional ideal if we bother to choose the δ_i to lie in $\mathrm{Der}_k(R,R)$, but this is an unimportant distinction. Indeed, we will end up working with fractional ideals during the proof of Theorem 15 anyway. By our definition, L describes a nonsingular foliation on the blowup $\tilde{V} = \mathrm{Bl}_J(V)$ if and only if the sheaf

$$\overline{\Omega_{\tilde{V}/k}} = \mathrm{Image}(\Omega_{\tilde{V}/k} \xrightarrow{\phi} \hat{L})$$

is locally free on $\tilde{V}$, where ϕ is the map sending a generating section dx to the function

$$\begin{aligned} L &\longrightarrow K \\ \delta &\longmapsto \delta(x). \end{aligned}$$

Moreover we have proven in Corollary 7 that L does define a nonsingular foliation on $\tilde{V}$ if and only if there is a fractional ideal S for R and a number α such that

$$S\mathscr{J}(J) = J^\alpha.$$

The most important case of this is when $L = \mathrm{Der}_k(K,K)$, in which case L defines a nonsingular foliation on $\tilde{V}$ if and only if $\tilde{V}$ is nonsingular.

THEOREM 15. (i) *There is always an inclusion* $\mathscr{J}(J)^{r+2} \subset \mathscr{J}(J\mathscr{J}(J))$.

(ii) *If* L *lifts to a nonsingular foliation on the blowup* $\tilde{V} = \mathrm{Bl}_J(V)$, *then there is an* N *such that the inclusion becomes an equality after both sides are multiplied by* J^N; *that is,*

$$J^N\mathscr{J}(J)^{r+2} = J^N\mathscr{J}(J\mathscr{J}(J)).$$

(iii) *Suppose conversely that* J *is any ideal such that the inclusion in* (i) *becomes* *an equality as in* (ii). *Then, letting* $I = J\mathscr{J}(J)$*, we find that* L *lifts to a nonsingular foliation on the blowup* $\mathrm{Bl}_I(V)$*, which is the same as the Gauss blowup* $\widetilde{\mathrm{Bl}_J(V)}$ *of the variety* $\mathrm{Bl}_J(V)$ *along* L*.*

Proof. For part (i), we have

$$\mathscr{J}(J)^{r+2} = \mathscr{J}(J)^{r+1}\mathscr{J}(J).$$

By Theorem 12, we have

$$\mathscr{J}(J)^{r+1}\mathscr{J}(J) \subset \mathscr{J}(J\mathscr{J}(J)).$$

Combining these gives the result.

Now for the proof of (ii). Suppose that L lifts to a nonsingular foliation on $\mathrm{Bl}_J(V)$. By Corollary 7, this means that, there is a fractional ideal S and a number α so that

$$\mathscr{J}(J)S = J^{\alpha}.$$

Now we have by Theorem 12

$$S^{r+1}\mathscr{J}(J\mathscr{J}(J)) \subset \mathscr{J}(SJ\mathscr{J}(J)).$$

Combining these, we see that

$$S^{r+1}\mathscr{J}(J\mathscr{J}(J)) \subset \mathscr{J}(J^{\alpha+1}).$$

Using Theorem 14, we have

$$\mathscr{J}(J^{\alpha+1}) = J^{(r+1)\alpha}\mathscr{J}(J).$$

Combining the last two formulas and multiplying through by $\mathscr{J}(J)^{r+1}$ gives

$$(\mathscr{J}(J)S)^{r+1}\mathscr{J}(J\mathscr{J}(J)) \subset J^{(r+1)\alpha}\mathscr{J}(J)^{r+2}.$$

Again applying the result of Corollary 7, we see that $(\mathscr{J}(J)S) = J^{\alpha}$. Substituting this in the left-hand side of the displayed equation gives

$$J^{(r+1)\alpha}\mathscr{J}(J\mathscr{J}(J)) \subset J^{(r+1)\alpha}\mathscr{J}(J)^{r+2}.$$

Setting $N = (r+1)\alpha$ gives one the desired inclusion of ideals; the opposite inclusion is part (i), which has already been proven, multiplied by J^N. The combination gives the equality of ideals

$$J^N\mathscr{J}(J\mathscr{J}(J)) = J^N\mathscr{J}(J)^{r+2}.$$

This is a second time in the paper that two unrelated arguments give upper and lower bounds for an ideal, and the bounds match exactly.

Now for the proof of part (iii). Suppose that the equality above holds. Let $I = J\mathscr{J}(J)$. We can assume that $N \geqslant r+2$ and let $\beta = N - r - 2$. Multiplying both sides of the formula by $\mathscr{J}(J)^{\beta}$, we have

$$\mathscr{J}(J)^{\beta}J^N\mathscr{J}(I) = J^N\mathscr{J}(J)^{\beta+r+2} = (J\mathscr{J}(J))^N = I^N.$$

Letting $S = \mathscr{J}(J)^{\beta}J^N$, we have

$$S\mathscr{J}(I) = I^N.$$

By Corollary 7, this proves that L lifts to a nonsingular foliation on $\mathrm{Bl}_I(V)$. Finally, identify $\mathrm{Bl}_I(V)$ with the Gauss blowup of $\mathrm{Bl}_J(V)$ along L by Lemma 4. $\square$

8. *Connection with the Nash resolution question*

Let us connect Theorem 15, in the case of the unique codimension zero foliation L, with the Nash question. Recall that V is affine irreducible over k a field of characteristic zero, K the function field of V, and R its coordinate ring. For $L = \mathrm{Der}_k(K, K)$, we let $\delta_1, \ldots, \delta_r$ be a K basis of L, where we can assume that the δ_i lie in $\mathrm{Der}_k(R, R)$. We defined for each ideal J of R a new ideal $\mathscr{J}(J)$ of R with the property that $\mathscr{J}(J)$ is a fractional ideal divisor of a power of J if and only if L lifts to a nonsingular foliation on $\tilde{V} = \mathrm{Bl}_J(V)$. Moreover, by Proposition 1, since L is the unique codimension zero foliation, this happens if and only if $\mathrm{Bl}_I(V)$ is nonsingular.

Recall by Lemma 4 that the blowup of the product $J\mathscr{J}(J)$ is the same as the result of blowing up J to get $\tilde{V}$ and then blowing up the highest exterior power of the reduced differentials of $\tilde{V}$.

Theorem 15, the main theorem of Section 7, states in this situation that when $\tilde{V}$ is nonsingular, so that the second blowup is an isomorphism, then the inclusion $J^N\mathscr{J}(J)^{r+2} \subset J^N\mathscr{J}(J\mathscr{J}(J))$ becomes an equality for some N, and that when this equality does hold, then the result of blowing up the highest exterior power of the reduced differentials of $\tilde{V}$ is nonsingular.

One can consider a chain of ideals

$$\begin{aligned} J_0 &= R \\ J_1 &= \mathscr{J}(R) \\ J_2 &= \mathscr{J}(R)\mathscr{J}(\mathscr{J}(R)) \\ J_3 &= \mathscr{J}(R)\mathscr{J}(\mathscr{J}(R))\mathscr{J}(\mathscr{J}(R)\mathscr{J}(\mathscr{J}(R))) \\ &\ \ \cdots \\ J_{i+1} &= J_i\mathscr{J}(J_i). \end{aligned}$$

The result of blowing up J_i is the same as starting with V and sequentially blowing up the highest exterior power of the reduced differentials to obtain a sequence of varieties $V_i \longrightarrow V_{i-1} \longrightarrow \ldots \longrightarrow V_0 = V$. Thus blowing up the ideal J_i accomplishes in one step what could otherwise be done in i steps of blowing up the highest exterior power of the reduced differentials:

$$V_i = \mathrm{Bl}_{J_i} V.$$

The chain of blowups stops (with all higher blowups being isomorphisms) if and only if some J_i resolves the singularities of V. After this, the ideal classes of the higher ideals $J_{i+1}, J_{i+2}, \ldots$, which are clearly multiples of J_i, are also divisors of a power of J_i.

In this context, the result of Section 7 tells you how to check when you have successfully resolved V. It says that one need only check that the inclusion

$$J_i^N\mathscr{J}(J_i)^{r+2} \subset J_i^N\mathscr{J}(J_{i+1})$$

is an equality for some N. When J_i resolves, this condition holds, and when the condition holds, J_{i+1} resolves.

Since $L = \mathrm{Der}_k(K, K)$, the above sequence of blowups is just the sequence of 'Nash' blowups, and the Nash question asks whether they eventually resolve the singularities of V. Therefore, we have a completely explicit reformulation of the Nash question.

THEOREM 16. *The Nash question holds in the affirmative for V if and only if the inclusion above becomes an equality for some sufficiently large i and N.*

To obtain the formulation in Section 1, note that when the inclusion above is an equality, it remains so when both sides are multiplied by J_{i+1}, and apply the basic definitions.

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